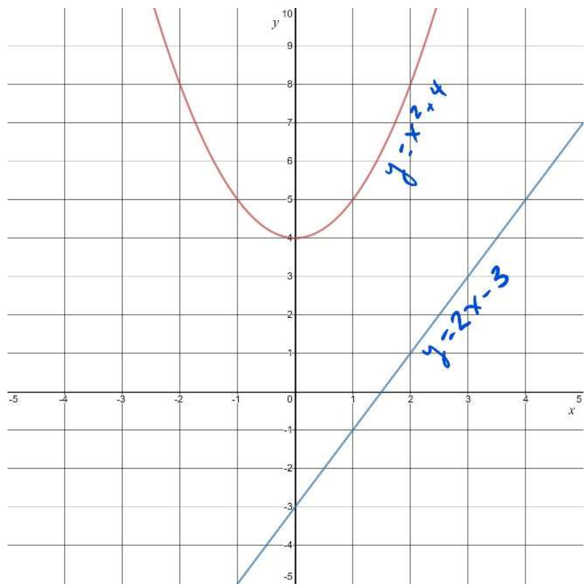
No Solution/Infinite Solutions

No Solution

If you think of the two equations as functions, they would be two functions that never intersect.

If they are both linear equations, they will have the same slope but different y-intercepts.

For example, if we had the system of equations $y = x^2 + 4$ and $y = 2x - 3$, we could graph them and see that they never intersect.



Here is an example of two linear equation that never intersect.

$$y = \frac{2}{3}x + 1$$

$$3y = 2x - 6$$

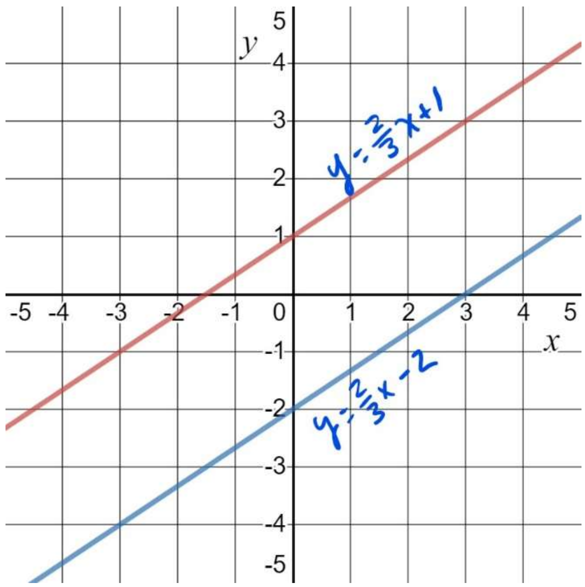
It isn't easy to immediately notice, but both equations have the same slope. Solve for y in the second equation.

$$\frac{3y}{3} = \frac{2x}{3} - \frac{6}{3}$$

$$y = \frac{2}{3}x - 2$$

$\uparrow$ slope $= 2/3$

Both lines have a slope of $\frac{2}{3}$. Here is a graph of the two lines.



Practice 1

In the system of linear equations below, *a* is a constant. If the system has no solution, what is the value of *a*?

$$\frac{1}{5}x - \frac{1}{10}y = 10$$

$$ax - 2y = 15$$

A) 4
B) 2
C) 3
D) 1

SUPER TIP
If the question asks for the number of real solutions, you may have to use the quadratic formula to see if the solutions are imaginary (not real). Also, if the greatest exponent is 2, it is only possible to have 2, 1 or 0 solutions.

Example

How many solutions are there to the equation $2x^2 - 4x = 9x + 1$?

A) 3
B) 2
C) 1
D) 0

In this case, you can immediately eliminate choice A, because there can't be 3 solutions if the greatest exponent is 2.

Since the greatest exponent is 2, we should get one side of the equation equal to zero then see if we can factor. If we can't factor, use the quadratic formula.

$$2x^2 - 4x = 9x + 1$$
$$-9x \quad -9x$$
$$2x^2 - 13x = 1$$
$$-1 \quad -1$$
$$2x^2 - 13x - 1 = 0$$

use the quadratic formula

$a = 2$, $b = -13$, $c = -1$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\frac{-(-13)\pm\sqrt{(-13)^2-4(2)(-1)}}{2(2)}$$

$$\frac{13\pm\sqrt{169+4}}{4}$$

$$\frac{13\pm\sqrt{173}}{4}$$

We can stop here. We don't have to find the exact solutions, just the number of real solutions.

There are two real solutions, they are:

$$\frac{13+\sqrt{174}}{4}\ and\ \frac{13-\sqrt{174}}{4}$$

The answer is B) 2.

Infinite Solutions

If you think of the two equations as functions, they would be graphed on top of each other.

If they are both linear equations, they will have the same slope and y-intercept.

For example, the following system of equations has infinite solutions.

$3y - 12x = 9$

$5y = 20x + 15$

solve for y in both equations

$$3y - 12x = 9 \qquad\qquad 5y = 20x + 15$$

$$\underline{\quad +12x \quad +12x} \qquad\qquad \frac{5y}{5} = \frac{20x}{5} + \frac{15}{5}$$

$$3y = 12x + 9 \qquad\qquad y = 4x + 3$$

$$\frac{3y}{3} = \frac{12x}{3} + \frac{9}{3}$$

$$y = 4x + 3$$

same equation! so infinite solutions

There will be infinite solutions, if they two equations are multiples of each other.

For example, the following equations are multiples of each other and the system has infinite solutions.

$2x - y = 12$

$6x - 3y = 36$

$2x - y = 12$

same equation, so infinite solutions

$6x - 3y = 36$

$\frac{6x}{3} - \frac{3y}{3} = \frac{36}{3}$

$\rightarrow 2x - y = 12$

Practice 2

In the following system of equations, a is a constant. What is the value of a if the system of equations has infinite solutions?

$ax + 2y = 10$

$3y - 6x = -15$

A) - 4
B) -1
C) 3
D) 6

$2y = -ax + 10$

$y = -\frac{a}{2}x + 5$

$3y - 6x = -15$

$3y = 6x - 15$

$y = 2x - 5$

Set slopes equal

$-\frac{a}{2} = 2$

$a = -4$

More Practice

Practice 3

The system of equations $4x + 8y = 12$ and $ax + 6y = 9$ has an infinite number of solutions. What is the value of a?

3

Practice 4

The system of equations $6x + 9y = 14$ and $ay + bx = -3$ have no solutions. What must $\frac{b}{a}$ equal?

A) $\frac{3}{2}$
B) $\frac{2}{3}$
C) $\frac{7}{3}$
D) $\frac{3}{7}$

Answers to Practice Problems

Practice 1: A

Solve for y in both

$\frac{1}{5}x - \frac{1}{10}y = 10$

Least Common Denominator = 10

$10\left(\frac{1}{5}x - \frac{1}{10}y = 10\right)$

$2x - y = 100$

$-2x \qquad -2x$

$-y = -2x + 100$

$y = 2x - 100$

$ax - 2y = 15$

$-ax \qquad -ax$

$-2y = -ax + 15$

$\frac{-2y}{-2} = \frac{-ax}{-2} + \frac{15}{-2}$

$y = \frac{a}{2}x - 7.5$

No solution → Same Slope

$y = 2x - 10 \qquad y = \frac{a}{2}x - 7.5$

$2 = \frac{a}{2}$

$(2)2 = \frac{a}{\cancel{2}}(\cancel{2})$

$4 = a$

Practice 2: -4

Solve for y in both equations

$ax + 2y = 10 \qquad 3y - 6x = 15$

$-ax \quad -ax \qquad +6x \quad +6x$

$2y = -ax + 10 \qquad 3y = 6x + 15$

$\frac{2y}{2} = \frac{-ax}{2} + \frac{10}{2} \qquad \frac{3y}{3} = \frac{6x}{3} + \frac{15}{3}$

$y = \frac{-a}{2}x + 5 \qquad y = 2x + 5$

Infinite Solutions:

Same slope & same y-int.

These both have a y-int of 5.

The slopes should also equal.

$$\frac{-a}{2} = 2$$

$$\frac{-a}{2}(-2) = 2(-2)$$

$$a = -4$$

Practice 3: 3

Infinite number of solutions means the lines are the same.

Solve for y in both.

$$4x + 8y = 12$$

$$\underline{-4x} \qquad \underline{-4x}$$

$$8y = -4x + 12$$

$$\frac{\cancel{8}y}{\cancel{8}} = \frac{-4x}{8} + \frac{12}{8}$$

$$y = \frac{-1}{2}x + \frac{3}{2}$$

$$ax + 6y = 9$$

$$\underline{-ax} \qquad \underline{-ax}$$

$$6y = -ax + 9$$

$$\frac{6y}{6} = \frac{-ax}{6} + \frac{9}{6}$$

$$y = \frac{-a}{6}x + \frac{3}{2}$$

They both have a y-int of $-3/2$. Their slopes must match up also.

$$-\frac{1}{2} = -\frac{a}{6}$$

$$(-6)\frac{-1}{2} = \frac{-a}{6}(-6)$$

$$\boxed{3} = a$$

Practice 4: B

No solution → parallel lines
(same slope)
Solve for y in both equati

$$6x+9y=14 \qquad ay+bx=$$
$$-6x \quad -6x \qquad -bx\ -$$
$$9y=-6x+14 \qquad ay=-bx$$
$$\frac{\cancel{9}y}{\cancel{9}}=\frac{-6x}{9}+\frac{14}{9} \qquad \frac{\cancel{a}y}{\cancel{a}}=\frac{-bx}{a}$$
$$y=\frac{-2}{3}x+\frac{14}{9} \qquad y=-\frac{b}{a}x-$$

The slopes must equal, so

$$\frac{-2}{3}=-\frac{b}{a}$$

We have to find $\frac{b}{a}$, so
multiply both sides by -1

$$(-1)\left(-\frac{2}{3}\right)=\left(\frac{-b}{a}\right)(-1)$$
$$\boxed{\frac{2}{3}}=\frac{b}{a}$$

Percent

Fractions to Percentages

A percent represents the number you would expect out of 100.

If you are only correct 1 out of every 2 questions, then you would be correct 50 out of 100 times, or 50% of the time.

To change a fraction to a percent, you can create an equivalent fraction with a denominator of 100. The numerator is the percent.

$$\frac{3}{4} = \frac{75}{100} = 75\%$$

$$\frac{1}{10} = \frac{10}{100} = 10\%$$

$$\frac{2}{5} = \frac{40}{100} = 40\%$$

Practice 1

Convert the following fraction to a percent.

$$\frac{3}{20} = \frac{15}{100} = 15\%$$

Decimals to Percentages

To change a decimal to a percent, use the fact that 0.38 means 38 hundredths = $\frac{38}{100}$ = 38%, or simply move the decimal 2 places right.

0.45 = 45%, 0.05 = 5%, 0.4 =0.40 = 40%

Practice 2

Convert the following decimals to percentages.

$0.7 =$ 70%

$0.043 =$ 4.3%

$0.58 =$ 58%

$1.25 =$ 125%

Percent Formula

$$\frac{Percent}{100} = \frac{Part\ (is)}{Whole\ (of)}$$

Or we can find the percent of a number using the following equation:

$$Part = Percent\ (as\ a\ decimal) * Whole$$

For example. If we have the question "What is 30% of 60?" we can solve the questions using either method.

$$\frac{Percent}{100} = \frac{Part\ (is)}{Whole\ (of)}$$

$$\frac{30}{100} = \frac{x}{60}$$

$$1800 = 100x$$
$$18 = x$$

or

$$Part = Percent\ (as\ a\ decimal) * Whole$$

$$x = 0.30 * 60$$

$$x = 18$$

Practice 3
What is 20% of 15?

3

Practice 4
7 is what percent of 28?

25%

Practice 5

8 is 5% of what?

40

Tax, Mark-up, and Discount (Increase and Decrease)

Increase by a Percent

Tax and mark-up increase the cost of a purchase.

For example, if a pair of shoes is \$80 and there is 8% tax, we would have to add the tax on to the total cost.

Price: \$80

Tax: 8%

$$\begin{array}{r} \$80 \\ \times\ .08 \\ \hline 6.40 \end{array} \leftarrow \text{tax}$$

$$\begin{array}{r} \$80.00 \\ +\ 6.40 \\ \hline \$86.40 \end{array} \leftarrow \text{total}$$

Another way to look at it is that 100% of the price of the shoes is \$80.
If it increases by 8% because of tax, the cost is now 108% of \$80.

Price: \$80

Tax: 8% → 108%

108% = 1.08

$$\begin{array}{r} 1.08 \\ \times\ 80 \\ \hline 86.40 \end{array} \quad \$86.40$$

The same concept applies to all increases.

If something increases by 40%, it will be 140% of what it previously was.

If something increases by 7.5%, it will be 107.5% of what it previously was.

If something increases by 100%, it will be 200% of what it previously was.

Practice 6

If the cost of a dress is $\$130$, what will the total cost be after 7% tax? (Try to find the answer in one step)

$130(1.07) = 139.10$

Decrease by a Percent

Discount decreases the cost of a purchase.

For example, if a pair of shoes is $\$80$ and there is 30% discount, we would have to subtract the discount from the total cost.

Price: \$80
Discount: 30%

$$\begin{array}{r} 80 \\ \times\ .3 \\ \hline 24.0 \end{array} \leftarrow \text{Discount}$$

$$\begin{array}{r} 80 \\ -\ 24 \\ \hline 56 \end{array} \leftarrow \text{Sale Price}$$

Another way to look at it is that 100% of the price of the shoes is $\$80$.
If it decreases by 30% because of tax, the cost is now 70% of $\$80$.

Price: \$80
Discount: 30% → 70%
70% = .7

$$\begin{array}{r} 80 \\ \times\ .7 \\ \hline 56.0 \end{array} \qquad \boxed{\$56}$$

The same concept applies to all decreases.

If something decreases by 40%, it will be 60% of what it previously was.

If something decreases by 7.5%, it will be 92.5% of what it previously was.

If something decreases by 24%, it will be 76% of what it previously was.

Practice 7

If the cost of a tablet is \$840, what will the total cost be after 25% discount? (Try to find the answer in one step)

$840(.75) = 630$

Equations that Incorporate Percents

On the SAT, you may be asked to identify equations that can be used to solve for a variable. In other words, sometimes you don't have to solve for a missing value, you just need to choose the equations that can be used to solve for the missing value.

First, let's review some expressions.

If a book costs $\$x$ and there is 5% tax, the total cost would be $\$1.05x$ because we have to increase the cost by 5%.

If a book costs $\$x$ and there is 15% discount, the total cost would be $\$.85x$ because we have to decrease the cost by 15%, which would leave us with 85% of the cost.

If a book costs $\$x$, but there is a 15% discount and 5% tax, the total cost would be $\$1.05(.85x)$.

Practice 8

James purchases movie tickets that cost \$12.95 each plus tax. The tax is equal to 9%, and James spends an additional untaxed amount of \$7.00 for snacks. Which of the following represents James' total charge in dollars, for *x* movie tickets?

A) $(12.95 + 0.09x) + 7$

B) $1.09(12.95 + 7)x$

C) $1.09(12.95x) + 7$

D) $1.09(12.95x + 7)$

Percent Change

The formula for percent change, whether increase or decrease, can be found with a simple equation.

$$percent\ change = \frac{change}{original} \times 100\%$$

For example, if a stock increased from \$80 to \$100, the change is \$20 and the original value was \$80. Let's use the formula to find the percent change.

$$percent\ change = \frac{change}{original} \times 100\%$$

$$percent\ change = \frac{20}{80} \times 100\% = 25\%$$

IF the stock decreased from \$100 to \$80, the change is still \$20, but this time the original value is \$100. Let's use the formula to find the percent change in this scenario.

$$percent\ change = \frac{change}{original} \times 100\%$$

$$percent\ change = \frac{20}{100} \times 100\% = 20\%$$

Practice 9

The data in the table shows the number of people in each profession for a fake town for the years 2009 and 2019. Which profession had the greatest percent increase from 2009 to 2019?

	2009	**2019**
Doctor	580	640
Lawyer	375	410
Engineer	605	590
Hockey Player	2	4

Hockey 100% increase

Answers to Practice Problems

Practice 1

$$\frac{3}{20} = \frac{15}{100} = 15\%$$

Practice 2

$0.7 = 0.70 = 70\%$

$0.043 = 0.043 = 4.3\%$

$0.58 = 0.58 = 58\%$

$1.25 = 1.25 = 125\%$

Practice 3

20% of 15

20% × 15

20% = 0.2

$$\begin{array}{r} 15 \\ \times .2 \\ \hline 3.0 \end{array}$$

Practice 4

7 is what percent of 28?

$$\frac{\%}{100} = \frac{\text{Part}}{\text{Whole}}$$

$$\frac{x}{100} = \frac{7}{28}$$

$$28x = 700$$

$$\frac{\cancel{28}x}{\cancel{28}} = \frac{700}{28}$$

$$x = 25$$

Practice 5

8 is 5% of what?

$$\frac{\%}{100} = \frac{\text{Part}}{\text{Whole}}$$

$$\frac{5}{100} = \frac{8}{x}$$

$$5x = 800$$

$$\frac{\cancel{5}x}{\cancel{5}} = \frac{800}{5}$$

$$x = 40$$

Practice 6

Price: \$130

Tax: 7%

$7\% \uparrow = 107\%$

$$\begin{array}{r} 130 \\ \times 1.07 \\ \hline 910 \\ 000 \\ 130 \\ \hline 139.10 \end{array}$$

$\boxed{\$139.10}$

Practice 7

Price: \$840

Discount: 25%

$25\% \downarrow = 75\%$

$$\begin{array}{r} 840 \\ \times .75 \\ \hline 4200 \\ 5880 \\ \hline 630.00 \end{array}$$

$\boxed{\$630}$

Practice 8

x movie tickets for \$12.95 each

\$7 for snacks (untaxed)

9% tax

Cost of movie tickets: $12.95x$

Cost of movie tickets with tax: $1.09(12.95x)$

Plus, \$7 for snacks:

$1.09(12.95x) + 7$

Practice 9

Doctor $580 \to 640$

$\frac{60}{580} \times 100\% \approx 10.3\%$

Lawyer $375 \to 410$

$\frac{35}{375} \times 100\% = 9.3\%$

Engineer $605 \to 590$

decreased

Hockey Player $2 \to 4$

$\frac{2}{2} \times 100\% = 100\%$

$\boxed{\text{Hockey Player}}$

Growth and Decay

To have a strong understanding of growth and decay, you need knowledge of percentages first.

If you haven't gone through the chapter on percent, I suggest you go back and review that chapter first.

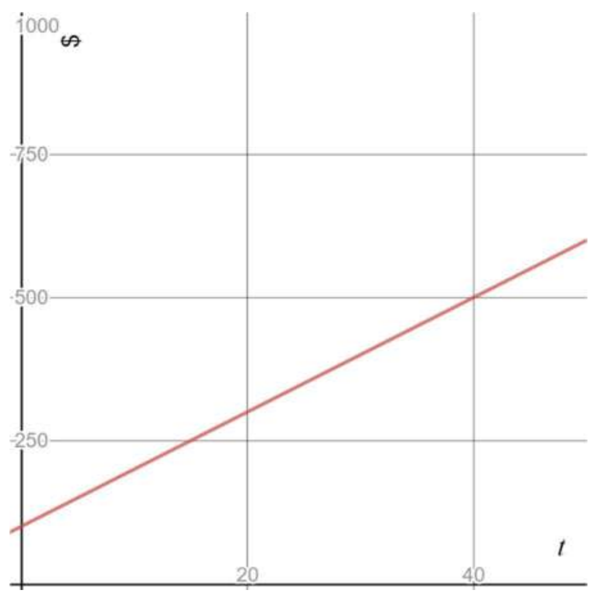
Exponential Growth

Exponential growth is typically caused by growth that is based on a percentage or ratio.

To better understand the difference, compare exponential growth to linear growth.

Suppose you deposited $100 in the bank and received $10 every year from the bank for keeping your money there. The amount of money in your account would increase linear because it would increase at a constant rate: $10/year.

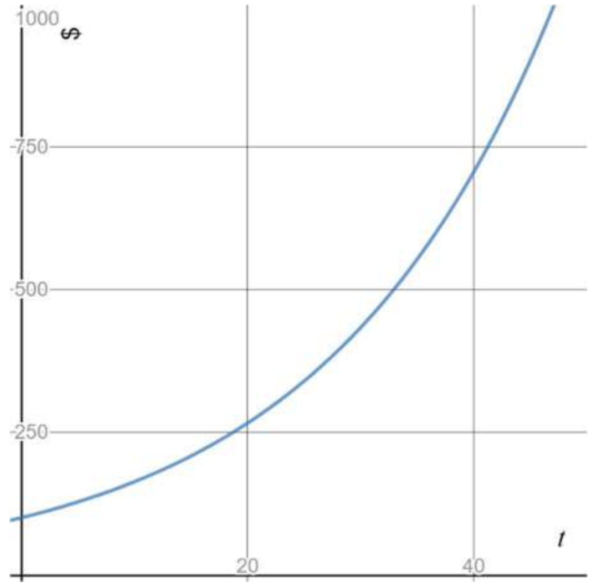
The amount of money in the account could be represented by the function $f(t) = 10t + 100$



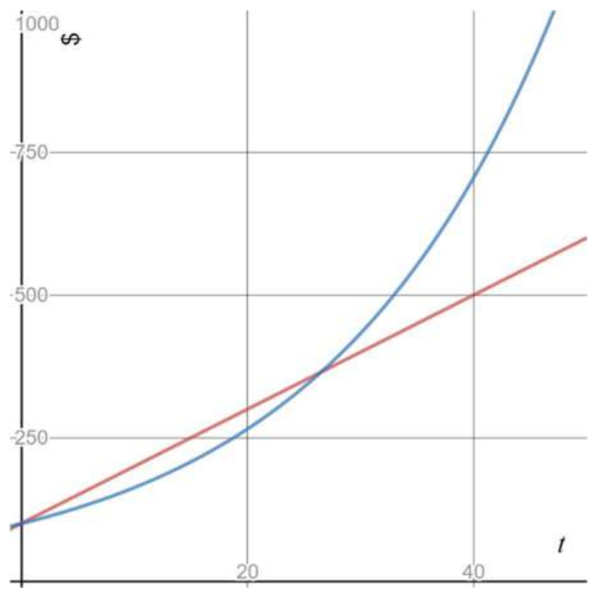
Suppose you deposited $100 in a different bank and received 5% interest each year. The first year you would only receive $5 because 5% of $100 is $5. However, the next year you would receive $5.25 because you would get 5% of $105. The following year, you would receive 5% of $110.25, which is $5.51. The amount of money you receive each year is bigger than the previous year's amount. This is exponential growth.

More precisely, since it is growing by 5%, each year will be 105% of the previous year, so each year will be a multiple of 1.05 of the previous year.

The amount of money in the account could be represented by the function $g(t) = 100(1.05)^t$



Here is a comparison of the two. As you can see from the graph, exponential growth results in a graph that is curved upward.



The function $g(t) = 100(1.05)^t$ is based on an initial value of \$100 that increases by 5% each year. It is based on the general formula:

$$A = A_0(1 + r)^t$$

where A is the value, A_0 is the initial value (in this case \$100), r is the rate (in this case $.05$ for 5%), and t is time (in this case in years)

Since we add the rate to one, we end up with growth.

Practice 1

This practice question has two parts.

Semande opened a savings account with $10,000. The savings account earns 3% annual interest. If Semande doesn't make any deposits or withdrawals, the value of his account could be found using the formula $V = 10{,}000(k)^t$.

Part 1

If t represents years, what number should be used for k?

1,03

Part 2

What would be the value of Semande's account in 10 years? Round to the nearest dollar.

13,439

Exponential Decay

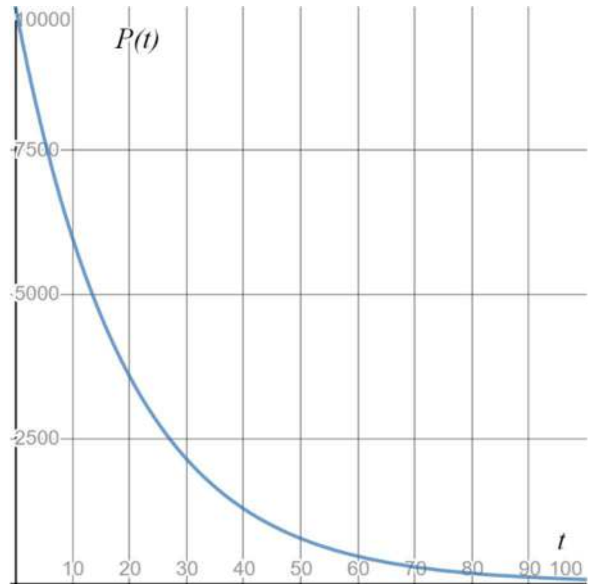
Like exponential growth, exponential decay is typically based on a percentage or ratio. However, since it is decaying (decreasing) the ratio is less than one.

Let's look at an example. Suppose the population of an organism was initially 10,000 and the population decreases by 5% each year. The first year, the population would decrease by 5% of 10,000, which is 500. The population next year would start at 9,500 and would decrease by 5% again that year, so it would decrease by 475. Each year, the population decreases by 5%.

More precisely, since it is decreasing by 5%, each year will be 95% of the previous year, so each year will be a multiple of 0.95 of the previous year.

The population of the organisms could be represented by the function $P(t) = 10{,}000(0.95)^t$

Here is a graph of $P(t)$.



The function $P(t) = 10{,}000(0.95)^t$ is based on an initial value of 10,000 that decreases by 5% each year. It is based on the general formula:

$$A = A_0(1 - r)^t$$

where A is the value, A_0 is the initial value (in this case 10,000), r is the rate (in this case $.05$ for 5%), and t is time (in this case in years)

Since we subtract the rate from one, we end up with decay.

Practice 2

The population of ducks at a particular park have been decreasing each year. A researcher found that the population decreases by 7% each year. If the population of the ducks 2 years ago was 2,400, what will the population of the ducks be in 5 years?

1,444

SUPER TIP
Sometimes it easier to do the math by hand, such as when a value doubles, halves, or triples.

Example

Since 2000, the population of deer in a town has doubled. If the population in 2003 was 9,920, what was the deer population in the town in 2001?

Population doubled each year, so to find the population of the previous year we just halve the number.

2003 : 9,920
2002 : 4,960
2001 : $\boxed{2,480}$

More Practice

Practice 3

An investment is projected in increase by 6% each year. If a $10,000 investment is made, how much will it be worth in 5 years? (Round your answer to the nearest dollar)

$10{,}000(1.06)^5 = 13{,}382$

Practice 4

A company has spent $24,000 on equipment. The accountant predicts that the equipment will decrease in value by 18% each year for the next 10 years. How much will the equipment be worth in 4 years? (Round your answer to the nearest dollar)

$24000(.82)^4 = 10{,}851$

Practice 5

The number of subscribers to a local newspaper has halved each year for the past 4 years. If there are currently 120,000 subscribers, how many subscribers did the newspaper have two years ago?

Now 120
last 240
2yrs 480

480,000

Answers to Practice Problems

Practice 1

Part 1

increase of 3% → 103%

$103\% = \boxed{1.03}$

OR

$A = A_0(1+r)^t$

$A = A_0(1+.03)^t$

$A = A_0(1.03)^t$

$\boxed{1.03}$

Part2

$V = 10{,}000(1.03)^t$

$t = 10$

$V = 10{,}000(1.03)^{10}$

$V = 13{,}439.163$

$\boxed{\$13{,}439}$

Practice 2: 1,444

2 years ago: 2,400 ducks

decrease by 7% → 93%

$P = 2{,}400(.93)^t$

2,400 was 2 years ago

To find P 5 years from now, we need to use $t = 7$

$P = 2{,}400(.93)^7$

$P = \boxed{1{,}444}$

Practice 3: $13,382

6% increase = 1.06

$10{,}000(1.06)^5$

$13{,}382.256$

$\boxed{\$13{,}382}$

Practice 4: \$10,851

18% decrease → 82%

$24{,}000(.82)^4$

10,850.922

$\boxed{\$10{,}851}$

Practice 5: 480,000

Subscribers halved each year,
so the previous year would
be double.

This year: 120,000

Previous year: 240,000

2 years ago: $\boxed{480{,}000}$

Exponents

Multiplying Terms with Exponents

When you multiply terms that have the same base, you add exponents.

$$x^2 \times x^5 = x^7$$

Here is a more complicated one

$$x^2y^3z \times xy^5z^4$$

If there is no exponent written, there is an exponent of 1.

$$(x^2y^3z)(xy^5z^4)$$

$$x^3 \quad y^8 \quad z^5$$

Practice 1

Simplify $(a^3b^2c^5)(a^2c^4)$

$$a^5b^2c^9$$

Dividing Terms with Exponents

When you divide terms that have the same base, you subtract exponents.

$$\frac{x^8}{x^3} = x^5$$

Here is a more complicated one

$$\frac{x^5y^3z^4}{x^2y^3z}$$

The y^3 terms in the numerator and denominator cancel out.

$$\frac{x^5\cancel{y^3}z^4}{x^2\cancel{y^3}z} = x^3z^3$$

Practice 2

Simplify the expression below.

$\frac{a^5b^7c^3}{a^3bc^2}$ a^2b^6c

Raising an Expression with Exponents to Another Power

When you raise an expression with exponents to another power, you multiply the exponents.

Example

Simplify the expression below.

$(a^3b^2c^8)^3$

$(a^3b^2c^8)^3$

multiply each exponent by 3

$a^9b^6c^{24}$

Example

Simplify the expression below.

$(3a^4b^7)^2$

$(3a^4b^7)^2$

Raise everything to the second power.

3 gets squared.

Multiply the exponents by 2.

$9a^8b^{14}$

Practice 3

Simplify the expression below.

$(3x^2y^5z)^4$

$81x^8y^{20}z^4$

Exponent of Zero

Any number raised to zero is 1.

$$x^0 = 1$$

Also

$$\left(\frac{x^2y^3 - 32}{4x^5 - 3z^2}\right)^0 = 1$$

There are a lot of terms, but the entire expression is raised to 0, so it is equal to 1.

Be careful.

$$xy^0 = x(1) = x$$

In this example, only the y is raised to 0, so y^0 becomes 1. The x remains x.

Practice 4

Simplify the expression below.

$$\left(\frac{x^4y^2z^{12}}{4x^{15}y^2 - 5x}\right)^0 \quad = 1$$

Negative Exponents

Think of a negative exponent as a sign that the term is in the wrong spot. If it is in the numerator, move it to the denominator and make the exponent positive. If it is in the denominator, move it to the numerator and make it negative.

For example

$$\frac{x^2y^3}{a^{-4}b^2}$$

In the example above, the exponent for the a is negative, so move it to the numerator.

$$\frac{a^4x^2y^3}{b^2}$$

Here is another example

$$\frac{2x^{-4}y^2}{z^3}$$

$$\frac{2x^{-4}y^2}{z^3}$$

Only the x has a negative exponent, so only the x should move to the denominator with its exponent. Do not move the coefficient, 2. Moving the 2 along with the x is a common mistake.

We would end up with

$$\frac{2y^2}{x^4z^3}$$

Example

Write the following with only positive exponents

$$\frac{5x^3y^{-4}z^{-2}}{-2a^5b^{-3}}$$

MOVE

$$\frac{5x^3 y^{-4} z^{-2}}{-2a^5 b^{-3}}$$

A negative coefficient is OK

MOVE

We end up with

$$\frac{5x^3b^3}{-2a^5y^4z^2}$$

Practice 5

Simplify so there are no negative exponents.

$$\frac{-3x^4y^{-2}}{x^2y^3}$$

$$\frac{-3x^4}{x^2y^3y^2} = -3x^2y^5$$

Fractions as Exponents

A fraction in the exponent represents a root, such as a square root or cubed root.

$$x^{\frac{1}{2}} = \sqrt{x}$$

$$x^{\frac{1}{3}} = \sqrt[3]{x}$$

$$x^{\frac{1}{4}} = \sqrt[4]{x}$$

The denominator of the exponent represents the root.

It is possible for the numerator to be a number other than 1. Treat the numerator like a regular exponent.

$$x^{\frac{4}{3}} = \sqrt[3]{x^4}$$

The 4 in the numerator means to the 4th power. The 3 in the denominator means cubed root.

Example

Rewrite $x^{\frac{5}{2}}$ with an integer power and root.

$x^{\frac{5}{2}}$ → 5th power
→ square root

$\sqrt{x^5}$

Practice 6

Rewrite the following as an expression without a fraction in the exponent.

$y^{\frac{4}{7}}$

$\sqrt[7]{y^4}$

Questions on the SAT will combine many of the rules together in a question.

Practice 7

Simplify the expression below.

$$\left(\frac{3x^2y^{-3}z^4}{x^{-3}y^2z}\right)^2$$

$$\frac{9x^4y^{-6}z^8}{x^{-6}y^4z^2} = \frac{9x^{10}z^6}{y^{10}}$$

Sometimes, the exponent questions will incorporate other skills too.

Practice 8

$$\frac{2^{3x}}{2^{3y}} = 2^{3x-3y} = 2^{3(x-y)} = 2^{3(3)} = 2^9 = 512$$

What is the value of the expression above if $x - y = 3$?

More Practice

Practice 9

$$\frac{y^{-3}\sqrt{x}}{x^{-2}\sqrt[3]{y}}$$

Which of the following is equivalent to the expression above?

A) $\frac{2x}{3y}$
B) $\frac{x}{y}$
C) $\frac{x^2\sqrt{x}}{y^3\sqrt[3]{y}}$
D) x^2y^3

Practice 10

$$\left(\frac{x^a}{x^b}\right)^2$$

If the expression above equals 1, what does $a-b$ equal?

$x^{2a-2b} = 1$

$2a-2b = 0$

$a-b=0$

$\boxed{0}$

Practice 11

$$\frac{2^{x^2}}{2^{y^2}}$$

If $x+y=2$ and $x-y=3$, what is the value of the expression above?

$2^{x^2-y^2}$

$2^{(x+y)(x-y)}$

$2^{(2)(3)} = 2^6 = 64$

Answers to Practice Problems

Practice 1: $a^5b^2c^9$

$$(a^3b^2c^5)(a^2c^4)$$

$$a^5 \quad b^2 \quad c^9$$

Practice 2: a^2b^6c

$$\frac{a^5b^7c^3}{a^3bc^2}$$

Subtract exponents

$$a^{5-3} \quad b^{7-1} \quad c^{3-2}$$

$$\boxed{a^2b^6c}$$

Practice 3: $81x^8y^{20}z^4$

$$(3x^2y^5z)^4$$

Raise each term to the 4th

$$3^4 = 3\cdot3\cdot3\cdot3 = 81$$

$$(x^2)^4 = x^8$$

$$(y^5)^4 = y^{20}$$

$$(z)^4 = z^4$$

$$\boxed{81x^8y^{20}z^4}$$

Practice 4: 1

The entire expression is raised to 0, so it becomes 1.

Practice 5: $-\frac{3x^6}{y^5}$

$$\frac{-3x^4y^{-2}}{x^{-2}y^3}$$

$$\frac{-3x^4x^2}{y^3y^2}$$

$$\frac{-3x^4x^2}{y^3y^2}$$ add exponents add exponents

$$\boxed{\frac{-3x^6}{y^5}}$$

Practice 6: $\sqrt[7]{y^4}$

$y^{4/7}$ power root

$\sqrt[7]{y^4}$

Practice 7: $\frac{9x^{10}z^6}{y^{10}}$

$$\left(\frac{3x^2y^{-3}z^4}{x^{-3}y^2z}\right)^2$$

Square each term.

$$\frac{9x^4y^{-6}z^8}{x^{-6}y^4z^2}$$

$$\frac{9x^4y^{-6}z^8}{x^{-6}y^4z^2}$$

$$\frac{9x^4x^6z^8}{y^4y^6z^2}$$

$$\frac{9x^{10}z^8}{y^{10}z^2}$$

$$\frac{9x^{10}z^8}{y^{10}z^2}$$

Can be reduced $8-2=6$

$$\frac{9x^{10}z^6}{y^{10}}$$

Practice 8: 512

$x - y = 3$

$\dfrac{2^{3x}}{2^{3y}}$ ← Dividing, so subtract exponents

2^{3x-3y}

$2^{3(x-y)}$

$2^{3(3)}$

2^{9}

$\boxed{512}$

Practice 9: C

$\dfrac{y^{-3}\sqrt{x}}{x^{-2}\sqrt[3]{y}}$

$\boxed{\dfrac{x^{2}\sqrt{x}}{y^{3}\sqrt[3]{y}}}$

Practice 10: 0

$\left(\dfrac{x^{a}}{x^{b}}\right)^{2} = 1$, $a - b = ?$

$\left(\dfrac{x^{a}}{x^{b}}\right)^{2} = 1$

$\dfrac{x^{2a}}{x^{2b}} = 1$

$x^{2a-2b} = 1$

0 $(x^{0} = 1)$

$x^{2a-2b} = 1$

$2a - 2b = 0$

$\dfrac{\cancel{2}a}{\cancel{2}} - \dfrac{\cancel{2}b}{\cancel{2}} = 0$

$a - b = \boxed{0}$

Practice 11: 64

$x+y=2\ ,\ x-y=3$

$\frac{2^{x^2}}{2^{y^2}}$ ← dividing, so subtract exponents

☆ $x^2-y^2=(x-y)(x+y)$

$x^2-y^2=(3)(2)$

$x^2-y^2=6$

$2^{x^2-y^2}$

$2^{x^2-y^2}=2^6=\boxed{64}$

Imaginary Numbers

Imaginary number, i

$\sqrt{-1}$ does not result in a real number. There is no real number that when multiplied by itself will result in -1.

$\sqrt{-1}$ is imaginary. We represent this imaginary number with the letter i.

$$\sqrt{-1} = i$$

Combining Like Terms with i

When combining like terms, treat i as if it's a variable.

For example, 3 and $3i$ are not like terms but $12i$ and $-3i$ are, just like 3 and $3x$ are not like terms, but $12x$ and $-3x$ are.

When combining like terms, just add the coefficients.

A simplified expression is typically in $a + bi$ form, where the number is written first followed by the i-term.

Example

Simplify $3 + 2i - 8i + 7$.

Here we will combine like terms by adding the like terms together. 3 and 7 are like terms, so we can add them together to get 10. $2i$ and $-8i$ are like terms, so we can add them together to get $-6i$.

We end up with $3 - 6i$, which is in $a + bi$ form. In this case, a is 3 and b is -6.

Example

Simplify $4xi + 12i - 6xi + 4$.

Here we will combine like terms by adding the like terms together. $4xi$ and $-6xi$ are like terms, so we can add them together to get $-2xi$.

Those are the only like terms, so we end up with $-2xi + 12i + 4$. Notice that we cannot write this $a + bi$ form since there is an x-term.

Example

Simplify $(9i - 7) + (-2i - 3)$.

To simplify, we should first get distribute to get rid of the parenthesis.

There is nothing written in front of $(9i - 7)$, so we can simply drop the parenthesis.

In front of $(-2i - 3)$ is a plus sign. If we distribute positive 1 to the terms, they will stay the same. We can drop this set of parentheses also.

We end up with $9i - 7 - 2i - 3$.

Combine the $9i$ and $-2i$ by adding them to get $7i$.

Combine the -7 and -3 by adding them to get -10.

The final answer is $7i - 10$.

Example

Simplify $(9i - 7) - (-2i - 3)$.

To simplify, we should first get distribute to get rid of the parenthesis.

There is nothing written in front of $(9i - 7)$, so we can simply drop the parenthesis.

In front of $(-2i - 3)$ is a minus sign. If we distribute negative 1 to the terms, they will get negated, resulting in $2i + 3$.

We end up with $9i - 7 + 2i + 3$.

Combine the $9i$ and $2i$ by adding them to get $11i$.

Combine the -7 and 3 by adding them to get -4.

The final answer is $11i - 4$.

Practice 1

Simplify $2i + 4 - 3i - 12$.

$-i - 8$

$-8 - i$

Practice 2

Find the sum of $3 + 2i$ and $-7 + 6i$.

$-4 + 8i$

Practice 3

Find the difference of $6 + 2i$ and $9 - 3i$.

$6 + 2i - (9 - 3i)$

$-3 + 5i$

Multiplying and Simplifying Expression with i

Since $\sqrt{-1} = i$, then $i^2 = -1$.

When we multiply terms or expressions that have i, we may end up with terms that have i^2. It's important to remember that $i^2$is imply -1.

Example

Simplify $(2 - 5i)(3 + 2i)$.

To simplify this, we will have to distribute twice (FOIL).

$$2 \times 3 = 6$$

$$2 \times 2i = 4i$$

$$-5i \times 3 = -15i$$

$$-5i \times 2i = -10i^2$$

So we would end up with

$$6 + 4i - 15i - 10i^2$$

We an combine the like terms, $4i$ and $15i$, to get

$$6 - 11i - 10i^2$$

Remember that $i^2$is really -1, so we have

$$6 - 11i - 10(-1)$$

$$6 - 11i + 10$$

Now, we can combine like terms to get

$$16 - 11i$$

Practice 4

Simplify $(2 + i)(3 - 4i)$

$6 - 8i + 3i - 4i^2$

$6 - 5i + 4 = \boxed{10 - 5i}$

Practice 5

Simplify $(5 - 2i)^2$

$(5 - 2i)(5 - 2i)$

$25 - 10i - 10i + 4i^2$

$25 - 20i - 4$

$\boxed{21 - 20i}$

Practice 6

Simplify $(4 + 2i)(4 - 2i)$

$16 - 8i + 8i - 4i^2$

$16 + 4$ $\boxed{20}$

Rationalizing the Denominator When the Denominator has an i-term

Occasionally, you may see a question that requires rationalizing the denominator.

In this case, the denominator has an expression including i.

To rationalize the following, we would simply multiply the number and denominator by i.

$$\frac{3 + 2i}{4i}$$

$$\frac{i(3 + 2i)}{i(4i)}$$

$$\frac{3i + 2i^2}{4i^2}$$

Remember that $i^2$is -1

$$\frac{3i + 2(-1)}{4(-1)}$$

$$\frac{3i - 2}{-4}$$

We typically wouldn't write a negative number in the denominator, so we should negate the numerator and denominator.

Our final answer is

$$\frac{-3i + 2}{4}$$

If we have to write it in $a + bi$ form, we would divide both terms in the numerator by the denominator to get

$$\frac{-3}{4}i + \frac{1}{2}$$

And write the i-term second

$$\frac{1}{2} - \frac{3}{4}i$$

Practice 7

Simplify the following

$$\frac{9 - i}{3i}\frac{(i)}{(i)} \qquad \frac{9i - i^2}{3i^2} = \frac{9i + 1}{-3} = \boxed{\frac{-9i - 1}{3}}$$

Rationalizing the Denominator When the Denominator has a Binomial with an i-term

Sometimes, the denominator is more complex, such as in the example below

$$\frac{3 + 4i}{4 - 61}$$

In this example, rationalizing the denominator is a little more difficult, but not impossible.

We have to multiply the numerator and denominator by something called the conjugate. The name doesn't matter, don't worry about memorizing the name.

The denominator is $4 - 6i$, so the conjugate is $4 + 6i$. The conjugate is just the same expression with the opposite sign between the two terms.

For example, if the denominator was $24+2i$, the conjugate would simply be $24-2i$.

Let's rationalize the denominator in the below expression

$$\frac{3+4i}{4-6i}$$

Multiply the numerator and denominator by $(4+6i)$

$$\frac{(3+4i)(4+6i)}{(4-6i)(4+6i)}$$

Remember that we have to distribute (FOIL). We end up with

$$\frac{12+18i+16i+24i^2}{16+24i-24i-36i^2}$$

When we combine like terms, the $i-terms$ in the deonominator ($24i$ and $-24i$) cancel out.

$$\frac{12+34i+24i^2}{16-36i^2}$$

Remember that $i^2$is -1, so it becomes

$$\frac{12+34i-24}{16+36}$$

$$\frac{-12+34i}{52}$$

All of the terms can be divided by 2, so we can reduce it to

$$\frac{-6+17i}{26}$$

The above can be our final answer, or if the question want the answer in $a+bi$ form, we would split up the fraction and reduce further if we can.

$$\frac{-6+17i}{26}=-\frac{6}{26}+\frac{17}{26}i=-\frac{3}{13}+\frac{17}{26}i$$

Our final answer would be

$$-\frac{3}{13}+\frac{17}{26}i$$

Practice 8

Simplify

$$\frac{5-3i}{2+2i}\frac{(2-2i)}{(2-2i)}$$

$$\frac{10-10i-6i+6i^2}{4-4i+4i-4i^2}=\frac{10-16i-6}{8}=\frac{4-16i}{8}=\boxed{\frac{1-4i}{2}}$$

Practice 9

Simplify

$$\frac{2+i}{1-i}\frac{(1+i)}{(1+i)} = \frac{2+2i+i+i^2}{1+i-i-i^2} = \frac{2+3i-1}{1+1} = \boxed{\frac{1+3i}{2}}$$

More Practice

Practice 10

Simplify $(3 - 2i) + (4 - 5i)$

$$\boxed{7 - 7i}$$

Practice 11

Simplify $(3 - 2i)(4 - 5i)$

$$12 - 5i - 8i + 10i^2$$

$$12 - 13i - 10$$

$$\boxed{2 - 13i}$$

Practice 12

Simplify

$$\frac{4 - i}{3 + 2i} \cdot \frac{(3 - 2i)}{(3 - 2i)} \qquad \frac{12 - 8i - 3i + 2i^2}{9 - 6i + 6i - 4i^2} = \frac{12 - 11i - 2}{13}$$

$$\boxed{\frac{10 - 11i}{13}}$$

Answers to Practice Problems

Practice 1: $-8 - i$

Simplify $2i + 4 - 3i - 12$.

$$2i + 4 - 3i - 12 = -8 - i$$

Practice 2: $-4 + 8i$

Find the sum of $3 + 2i$ and $-7 + 6i$.

Sum is the answer to an addition problem, so we should add these expressions together.

$$(3 + 2i) + (-7 + 6i)$$

$$3 + 2i - 7 + 6i$$

$$-4 + 8i$$

Practice 3: $-3 + 5i$

Find the difference of $6 + 2i$ and $9 - 3i$.

Difference is the answer to a subtraction problem, so we should subtract the expressions in the order they are written.

$$(6 + 2i) - (9 - 3i)$$

Distribute the negative to get

$$6 + 2i - 9 + 3i$$

$$-3 + 5i$$

Practice 4: $10 - 5i$

Simplify $(2 + i)(3 - 4i)$

$$(2 + i)(3 - 4i) = 6 - 8i + 3i - 4i^2$$

$i^2 = -1$ so we end up with

$$6 - 8i + 3i + 4$$

$$10 - 5i$$

Practice 5: $21 - 20i$

Simplify $(5 - 2i)^2$

$$(5 - 2i)(5 - 2i) = 25 - 10i - 10i + 4i^2$$

$i^2 = -1$ so we end up with

$$25 - 10i - 10i - 4$$

$$21 - 20i$$

Practice 6: 20

Simplify $(4 + 2i)(4 - 2i)$

$$(4 + 2i)(4 - 2i) = 16 - 8i + 8i - 4i^2$$

$i^2 = -1$ so we end up with

$$16 - 8i + 8i + 4$$

$$20$$

Practice 7: $\frac{-1-9i}{3}$

Simplify the following

$\frac{9-i}{3i}$

To rationalize this, we should multiply the numerator and denominator by i.

$$\frac{i(9-i)}{i(3i)} = \frac{9i-i^2}{3i^2}$$

$i^2 = -1$ so we end up with

$$\frac{9i+1}{-3}$$

We should not have a negative in the denominator, so negate the numerator and denominator to get

$$\frac{-9i-1}{3}$$

It is customary to write the i-term last, so we get

$$\frac{-1-9i}{3}$$

Practice 8: $\frac{1-4i}{2}$

Simplify

$\frac{5-3i}{2+2i}$

To rationalize this expression, multiply the numerator and denominator by the conjugate of $2+2i$, which is $2-2i$

$$\frac{(5-3i)(2-2i)}{(2+2i)(2-2i)} = \frac{10-10i-6i+6i^2}{4-4i+4i-4i^2}$$

$i^2 = -1$ so we end up with

$$\frac{10-16i-6}{4-4i+4i+4} = \frac{4-16i}{8}$$

We can simply further by dividing each term by 4

$$\frac{1-4i}{2}$$

Practice 9: $\frac{1+3i}{2}$

Simplify

$\frac{2+i}{1-i}$

To rationalize this expression, multiply the numerator and denominator by the conjugate of 1-i, which is $1+i$

$$\frac{(2+i)(1+i)}{(1-i)(1+i)} = \frac{2+2i+i+i^2}{1+i-i-i^2}$$

$i^2 = -1$, so we end up with

$$\frac{2+2i+i-1}{1+i-i+1} = \frac{1+3i}{2}$$

Practice 10: $7-7i$

$(3-2i)+(4-5i)$

This is just combining like terms

$3-2i+4-5i$

$\boxed{7-7i}$

Practice 11: $2 - 23i$

$(3-2i)(4-5i)$

$12 - 15i - 8i + 10i^2$

$12 - 15i - 8i + 10i^2$

$12 - 23i + 10i^2$

$i^2 = -1$

$12 - 23i + 10(-1)$

$12 - 23i - 10$

$\boxed{2 - 23i}$

Practice 12: $\frac{10-11i}{13}$

$\frac{4-i}{3+2i}$

To simplify, multiply the numerator and denominator by the conjugate of the denominator (change the middle sign)

$\frac{4-i\,(3-2i)}{3+2i\,(3-2i)}$

$\frac{(4-i)(3-2i)}{(3+2i)(3-2i)}$

$\frac{12 - 8i - 3i + 2i^2}{9 - 6i + 6i - 4i^2}$

$\frac{12 - 11i + 2(-1)}{9 - 4(-1)}$

$\frac{12 - 11i - 2}{9 + 4}$

$\frac{10 - 11i}{13}$